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Perfect fluids with less than extreme relativistic equation of state.

- Not very lucky, Difficult problem
- Interesting, rewards : Cosmology, colliding waves, star interiors.
- Experience with stat. axisymm $\begin{cases}\to \text{static, stationary} \\ \to \text{vacuum} \Rightarrow \text{Einstein - Maxwell}\end{cases}$
- Fluids are difficult.
- Two simple cases $\begin{cases}\to \text{Dust} \\ \to \epsilon = p.\end{cases}$
- Two spacelike or one spacelike, one timelike are different!
- Other solutions. Killing tensor, ~~Wahl~~ Wahlquist, $\epsilon + 3p = \text{const}$,
  About $\epsilon = p + k$, Kramer, easy to cheat you.
- I do not believe much in my luck.
- Chandra $\Rightarrow$ $\epsilon = p + k$, take surface orthogonal Killing fields.
- Repeat Weyl problem, I've got a Weyl-type family of solutions.
- Chandra's notation - Hydrodynamic equations - equation of state - stream potential.
- The Einstein equations - Gauge conditions - The vacuum case.
- $u_{(0)}$, $u_{(3)}$, ϵ, p all in terms of ϕ
- Why $\epsilon = p$ simply. Because $k = 0$!
- Gauge conditions for the present case: e^{β}, $e^{\mu_3 - \nu}$ separable.
- Redefinition of x^0, x^3 $\Rightarrow$ Five equations on transparency 6.
- Any weak point? Where the system link?
- Exactly the same as for the Weyl family.
- Different strategy $\to$ Universal conformal factor.
- One or three families?
- b, d_1, d_2 not free parameters, gauge, Simple solutions.
- Simple separable solutions for $\phi \sim \frac{\cos\frac{kx}{2}}{\cos x/2} \; \frac{\cos\frac{ky}{2}}{\cos y/2}$
- Singularities on null surfaces.
- Not continuously connected with $k = 0$.

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Perfect fluids with less than extreme relativistic equation of state.

I am not very lucky today, I have a difficult problem to overcome. I have to tell you, in the next hour or so, how to solve the Einstein equations, with two commuting Killing fields, when gravity is coupled with a perfect fluid satisfying the equation of state energy density (ϵ) = pressure (p) + const. (κ). Obviously the problem, the approach, the strategy, depends crucially on the details of the differential equations. And my big problem is how to present this work, without loosing you among my — too many and too complicated — equations. But not leave the room from the very beginning, I will try my best.

Since perfect fluids is not a very popular subject in this audience, as far as I know, I will elaborate a little bit about them.

The broad problem is to do something, to get control, to solve the Einstein – perfect fluid equations with two commuting Killing fields. The problem is interesting and the rewards of the potential success ~~are~~ outstanding: For one spacelike and one timelike Killing fields, such solutions represent star interiors. For two spacelike Killing fields represent inhomogeneous cosmological models or the interaction region of colliding gravitational and hydromagnetic plane waves. This last application coincides with my current interest and this is how I was motivated to go into the perfect fluids.

Before getting into the fluids lets pose for a moment and assess what we have leamt so forr, particularly

in the seventies, for spacetimes with two commuting Killing fields - about the stationary-axisymmetric problem.
(i) The static equations easy, the Weyl solutions, described by a linear - Laplace - equation. Rotation much more difficult (two orders of magnitude?) More sophisticated methods had to be used like the infinite dimensional group of transformations suggested by Geroch and completed by Kinnersley and the people around him. Also the inverse scattering method proved useful, Kramer - Neugebauer.
(ii) Second message: whatever can be achieved for the vacuum Einstein equations it can be easily extended to the Einstein-Maxwell electrovacuum equations. Although not obviously true for the P.D.E's, certainly applies to the transformation and the inverse scattering methods.

The first point I want to make is that although everything naturally extends from the vacuum to the Einstein-Maxwell equations, practically nothing extends to the perfect fluids (I am always considering two commuting Killing fields). Generally fluids are bad and rumor is - I've heard it from Chandra - that, while the Einstein and the Maxwell equations are God given, the fluid equations are man made.

Among the fluids there are two cases which are simple and completely understood
(i) For stationary axisymmetric spacetimes they are the dust solutions, $p=0$. The solution to the problem can be reduced to successive quadratures, achieved by Winicour (1975).
(ii) For two spacelike Killing fields, the extreme relativistic

fluids with $\epsilon = p$ (stiff equation of state) are easy. We have to solve the vacuum equations (~~Laplace~~ Weyl for surface orthogonal, Ernst equation for more generally) plus a linear hyperbolic equation for the fluids stream potential. The linear equation admits simple separable solutions and there is no additional difficulty than that required to solve the vacuum problem. For particular problems you might have to consider which vacuum background solutions or which solutions of the linear equations to choose but although is the exact treatment of a non-linear theory, it behaves like a linear problem. It splits to a background gravitational vacuum spacetime and a fluid propagating and affecting this background. The purely gravitational part of the background can be any vacuum solution.

This, the $\epsilon = p$ and two spacelike killing fields is the case we used with Chandra for studying the collision of gravitational and hydrodynamic waves which resulting in the transformation of null dust to a massive ~~f~~ extremely relativistic perfect fluid. We have to admit: We have chosen extremely relativistic perfect fluid in the interaction region because we could handle the equations.

(*) One remark, I've never seen written and took me sometime to realize: Contrary to the vacuum case, 2 spacelike Killing fields or one spacelike and one timelike are quite different cases. The reason: The fluids four velocity u^a, $T_{ab} = (\epsilon + p) u_a u_b - p g_{ab}$ is a timelike vector. For stationary axisymmetric, u^a ~~lies~~ lies in the two-plane spanned by the two Killing fields, where nothing is happening. ~~a~~ For two spacelike,

(4)

$$R_{ab} - \frac{1}{2} R g_{ab} + \Lambda g_{ab} = 0 \Leftrightarrow$$

$$R_{ab} = (\epsilon + p) u_a u_b + \frac{1}{2}(p - \epsilon + 2\Lambda) g_{ab}$$

u^a is orthogonal to the two Killing fields, it is tangent to the manifold of orbits, where all the action is.

The resulting equations are different in a few ~~p~~ but crucial terms. This is the reason why the two simplest cases are different.

Except from the above two cases, practically very little has been achieved so far for perfect fluids with two Killing fields. The standard pattern is to assume, except from the two Killing fields, a second rank Killing tensor as well. Then everything follows, including the equation of state. And to nobodies surprise, is usually ~~unnatur~~ aphysical, usually $\epsilon + 3p = $ const or $\epsilon + p = $ const, wrong sign. Although, very often, a third Killing field is implied.

Wahlquist, Phys. Rev. 172, 1291, 1968, type D, $\epsilon + 3p = $ const. Singularity free, pressure drops to zero → exterior, but wrong behavior ⦶, rigid rotation.

About $\epsilon = p + k$ Two months ago, Kramer published a solution with $\epsilon = p + k$ equation of state, and algebraically special Weyl tensor. He started from conditions on the Weyl tensor.

⊗ easy to cheat, $\Lambda = 0$ and $\epsilon = p \Leftrightarrow \Lambda \neq 0$ and $\epsilon = p + $ const. and forget to say that you have cosmological constant.

Well, I do not believe very much in my luck. So, I wanted to specify for the very beginning the equation of state and then try to see how much you can go. And in discussions with Chandra, after our $\epsilon = p$ solutions, ~~I~~ was easily convinced that the next to try should be perfect fluids with two spacelike Killing fields and $\epsilon = p + k$ equation of state. And since the real objective is to get experience on how to handle different equations of state and not how to handle rotation for the time being I will always

(5)

assume that the two Killing fields are surface orthogonal.

* So my problem was to repeat the Weyl program for $\epsilon = p + k$ fluids. As you will see, I will present a Weyl type family of solutions, characterized by the solutions of a linear hyperbolic equations. The only difference is that Weyl did his part 70 years ago!

* I will use the notation I've learnt from Chandra.

Metric of the form

$$ds^2 = e^{2\nu}(dx^0)^2 - e^{2\psi}(dx^1)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2,$$

$\partial/\partial x^1$, $\partial/\partial x^2$ the two Killing fields,

$$T^{ij} = (\epsilon + p) u^i u^j - p g^{ij}, \quad u^i u_i = 1, \quad (+ - - -),$$

ϵ, p, u^i also preserved by the two Killing fields.

* Hydrodynamic equations $\nabla^i T_{ij} = 0$.

Assume an equation of state but leave it ~~unsp~~ unspecified.

$\exists$ scalar, $\epsilon + p = f(\epsilon)$, set $\dot{f}_1 / f_1 = 1/f$.

$$u^{(0)} = u_{(0)} = e^{-\nu} u_0 \quad ; \quad u^{(3)} = -u_{(3)} = -e^{-\mu_3} u_3 .$$

Contracted $u^j(\nabla^i T_{ij})$ written as total divergence,

$$\left[e^{\psi + \mu_2 + \mu_3} u_{(0)} f_1\right]_{,0} - \left[e^{\nu + \psi + \mu_2} u_{(3)} f_1\right]_{,3} = 0$$

solve by a potential, the stream potential

$$e^{\psi + \mu_2 + \mu_3} u_{(0)} f_1 = \phi_{,3}$$
$$e^{\nu + \psi + \mu_2} u_{(3)} f_1 = \phi_{,0} .$$

Return to the uncontracted equation $\nabla^i T_{ij} = 0$, two but equivalent, only one independent, you substitute all the fluids quantities, u_a, ϵ for $\epsilon = p + k$ gives

$$\left[e^{\mu_3 - \mu_2 - \psi - \nu} \phi_{,0}\right]_{,0} - \left[e^{\nu - \psi - \mu_2 - \mu_3} \phi_{,3}\right]_{,3} = 0.$$

Then the characteristics of the fluid are algebraically determined from the stream potential ϕ:

$$- 2e \,\{ [e \ (e)_{,3}]_{,3} + [e \ (e)_{,0}]_{,0} \} +$$

$$+ e^{\nu - \mu_3} (\log x)^2 + e^{\mu_3 - \nu} (\log x)^2 - 4 (\epsilon + p)(u^2 + u^2) e^{\nu + \mu_3}$$

$$2\epsilon - k = e^{-2(\psi+\mu_2)}\left[\phi_{,3}^2 e^{-2\mu_3} - \phi_{,0}^2 e^{-2\nu}\right]$$

$$u_{(0)} = \frac{e^{-\psi-\mu_2-\mu_3}}{\sqrt{2\epsilon-k}}\,\phi_{,3}\ ; \quad u_{(3)} = \frac{e^{-\psi-\nu-\mu_2}}{\sqrt{2\epsilon-k}}\,\phi_{,0}.$$

Next we turn to the Einstein equations. We can impose one gauge condition involving (x^0, x^3) and one among x^1, x^2.

Notation $\beta = \psi + \mu_2$, $\chi = e^{\mu_2 - \psi}$. Equations

$$\left[e^{\mu_3-\nu}(e^\beta)_{,0}\right]_{,0} - \left[e^{\nu-\mu_3}(e^\beta)_{,3}\right]_{,3} = -2ke^{\beta+\nu+\mu_3}$$

$$\left[e^{\beta+\mu_3-\nu}(\log\chi)_{,0}\right]_{,0} - \left[e^{\beta+\nu-\mu_3}(\log\chi)_{,3}\right]_{,3} = 0$$

$$\beta_{,0}\boxed{(\nu+\mu_3)_{,3}} + \beta_{,3}\boxed{(\nu+\mu_3)_{,0}} + \left[\beta_{,0}(\nu-\mu_2)_{,3} - \beta_{,3}(\nu-\mu_3)_{,0} - 2\beta_{,03} - \beta_{,0}\beta_{,3}\right] =$$
$$= \frac{\chi_{,0}\chi_{,3}}{\chi^2} - 4(\epsilon+p)\,u_{(0)}u_{(3)}\,e^{\nu+\mu_3}$$

$$2\beta_{,3}e^{\nu-\mu_3}\boxed{(\nu+\mu_3)_{,3}} + 2\beta_{,0}e^{\mu_3-\nu}\boxed{(\nu+\mu_3)_{,0}} + e^{\nu-\mu_3}\left[\beta_{,3}^2 + 2\beta_{,3}(\nu-\mu_3)_{,3}\right] +$$
$$+ e^{\mu_3-\nu}\left[\beta_{,0}^2 + 2\beta_{,0}(\mu_3-\nu)_{,0}\right] =$$
$$= 2e^{-\beta}\left\{\left[e^{\nu-\mu_3}(e^\beta)_{,3}\right]_{,3} + \left[e^{\mu_3-\nu}(e^\beta)_{,0}\right]_{,0}\right\} +$$
$$+ e^{\nu-\mu_3}(\log\chi)_{,3}^2 + e^{\mu_3-\nu}(\log\chi)_{,0}^2 - 4(\epsilon+p)\left(u_{(0)}^2+u_{(3)}^2\right)e^{\nu+\mu_3}.$$

$$\left[e^{\mu_3-\mu_2-\psi-\nu}\phi_{,0}\right]_{,0} - \left[e^{\nu-\psi-\mu_2-\mu_3}\phi_{,3}\right]_{,3} = 0.$$

We can impose gauge conditions on $\beta = \psi + \mu_2$ and $\mu_3 - \nu$.

⑦

In vacuum and electrovacuum the gauge conditions are imposed compatibly with the first equation: The most usual choice for Kerr and other simple solutions,

$$e^{\beta} = \sqrt{(1-y^2)(1-\mu^2)} \quad , \quad "y = x^0", \quad "\mu = x^3", \qquad e^{\mu_3 - \nu} = \sqrt{1-y^2}$$

Two objectives achieved: (i) Gauge known from the very beginning
(ii) One of the Einstein equations is taken care of.

Then $(\log \chi)$ satisfies a linear equation and the remaining two equations will determine by quadratures the conformal factor $\nu + \mu_3$ in the manifold of orbits, the two-dimensional geometry orthogonal to the two Killing fields. And if the Killing fields are not hypersurface orthogonal, the strategy remains the same: Gauge $\rightarrow$ Ernst potential $\rightarrow$ conformal factor, Three separate steps.

⊗ You see what $k \neq 0$ makes. All three, gauge, norm of Killing field, fluid, conformal factor all coupled, not subproblems, all should be determined simultaneously. WE CANNOT FIX THE GAUGE FROM THE BEGINNING.

- One good news, ϵ, p, u_0, u_3 are not all unknowns, all expressed interms of ϕ alone.
- Why $k=0 \Leftrightarrow \epsilon = p$ simple. Gauge determined independently, $\log \chi$ determined independently, ϕ determined independently, it is exactly as for the vacuum case.

$$u(0) = \frac{}{\sqrt{\delta \phi_\mu^2 - \Delta \phi_\eta^2}} \qquad u(3) = \frac{}{\sqrt{\delta \phi_\mu^2 - \Delta \phi_\eta^2}}$$

For the gauge conditions I ask exactly the same thing as in the vacuum case, namely that e^{β} and $e^{\mu_3-\nu}$ are separable in x^0, and x^3;

$$e^{\beta} = A(x^0)\, B(x^3) \quad ; \quad e^{\mu_3-\nu} = \frac{\Gamma(x^0)}{E(x^3)}$$

(that one is product and the other ratio is just for later convenience). This are chosen compatibly with the gauge fixing equation in the vacuum and electrovacuum cases, compatibly with the entire system in the present case). The only way to try to convince you that it is not a restriction that I have is by saying that asking for separable solutions is not a restriction in the vacuum case.

~~I further specialize the~~ The above gauge conditions involve the relationship between (x^0, x^3) and (x^1, x^2). They do not ~~to~~ touch at all "what is x^0 and what is x^3". We make x^0 and x^3 ~~prolate~~ prolate spheroidal coordinates, i.e, the manifold of orbits of the form $(\ldots)\left[\frac{(d\eta)^2}{1-\eta^2} - \frac{(d\mu)^2}{1-\mu^2}\right]$, in which the Kerr and the colliding wave solutions are simple and from which I have experience with extensions. I set

$$\frac{dx^0}{\Gamma(x^0)} = \frac{d\eta}{\sqrt{1-\eta^2}} \quad , \quad \frac{dx^3}{E(x^3)} = \frac{d\mu}{\sqrt{1-\mu^2}} .$$

Gauge functions $A(\eta)$, $B(\mu)$, the combination $\Gamma E e^{\mu+\nu} = e^{2\Omega}$ appear in the conformal factor and nowhere else, metric

$$ds^2 = e^{2\Omega}\left[\frac{(d\eta)^2}{\Delta} - \frac{(d\mu)^2}{\delta}\right] - AB\left[\frac{(dx^1)^2}{\chi} + \chi\,(dx^2)^2\right],$$

fluid quantities

$$2\epsilon - k = \frac{e^{-2\Omega}}{A^2B^2}\left(\delta\, \Phi_{,\mu}^2 - \Delta\, \Phi_{,\eta}^2\right)$$

$$u_{(0)} = \frac{\Phi_{,\mu}\sqrt{\delta}}{\sqrt{\delta\Phi_{\mu}^2 - \Delta\Phi_{\eta}^2}} \qquad u_{(3)} = \frac{\Phi_{\eta}\sqrt{\Delta}}{\sqrt{\delta\Phi_{\mu}^2 - \Delta\Phi_{\eta}^2}}$$

In the equations we substitute the fluids quantities exclusively in terms of ϕ. We get

$$A\left[\frac{\sqrt{\Delta}}{A}\phi_{,\eta}\right]_{,\eta}\sqrt{\Delta}-B\left[\frac{\sqrt{\delta}}{B}\phi_{,\mu}\right]_{,\mu}\sqrt{\delta}=0$$

$$\frac{1}{A}\left[A\sqrt{\Delta}\,(\log\chi)_{,\eta}\right]_{,\eta}\sqrt{\Delta}-\frac{1}{B}\left[B\sqrt{\delta}\,(\log\chi)_{,\mu}\right]_{,\mu}\sqrt{\delta}=0$$

$$\frac{(\mathring{A}\sqrt{\Delta})^{\circ}\sqrt{\Delta}}{A}-\frac{(\mathring{B}\sqrt{\delta})^{\circ}\sqrt{\delta}}{B}=-2ke^{2\Omega}$$

$$\frac{2\mathring{A}}{A}\Omega_{,\mu}+\frac{2\mathring{B}}{B}\Omega_{,\eta}=\frac{\mathring{A}\mathring{B}}{AB}+\frac{\chi_{,\eta}\chi_{,\mu}}{\chi^2}-\frac{4\phi_{,\eta}\phi_{,\mu}}{A^2B^2}$$

$$\frac{4\mathring{A}\Delta}{A}\Omega_{,\eta}+\frac{4\mathring{B}\delta}{B}\Omega_{,\mu}=\Delta\left(\frac{2\mathring{\mathring{A}}}{A}-\frac{\mathring{A}^2}{A^2}\right)+\delta\left(\frac{2\mathring{\mathring{B}}}{B}-\frac{\mathring{B}^2}{B^2}\right)-2\left(\eta\frac{\mathring{A}}{A}+\mu\frac{\mathring{B}}{B}\right)+$$
$$+\frac{1}{\chi^2}\left(\Delta\chi_{,\eta}^2+\delta\chi_{,\mu}^2\right)-\frac{4}{A^2B^2}\left(\Delta\,\phi_{,\eta}^2+\delta\,\phi_{,\mu}^2\right).$$

weak point?, two

Five by five, what do you do? Here comes the crucial observation, which leads to the solution.

How do you solve equation for ϕ or $\log\chi$. They are total divergences

$$\Big[\underbrace{\frac{AB\sqrt{\Delta}}{\sqrt{\delta}}(\log\chi)_{,\eta}}_{G_{,\mu}}\Big]_{,\eta}=\Big[\underbrace{\frac{AB\sqrt{\delta}}{\sqrt{\Delta}}(\log\chi)_{,\mu}}_{G_{,\eta}}\Big]_{,\mu}$$

introduce potential, you think you have solve, you have not because you have to satisfy the integrability conditions, here

$$(\log\chi)_{,\eta}=\frac{\sqrt{\delta}}{AB\sqrt{\Delta}}G_{,\mu}\;;\qquad(\log\chi)_{,\mu}=\frac{\sqrt{\Delta}}{AB\sqrt{\delta}}G_{,\eta}\;\Rightarrow$$

$$\left(\frac{\sqrt{\Delta}}{AB\sqrt{\delta}}G_{,\eta}\right)_{,\eta}-\left(\frac{\sqrt{\delta}}{AB\sqrt{\Delta}}G_{,\mu}\right)_{,\mu}=0\iff A\left[\frac{\sqrt{\Delta}}{A}G_{,\eta}\right]_{,\eta}\sqrt{\Delta}-B\left[\frac{\sqrt{\delta}}{B}G_{,\mu}\right]_{,\mu}\sqrt{\delta}=0.$$

The equation for the potential. Hence, we could solve only one of the two linear equations and obtain the other by quadratures:

$$G = c\phi, \qquad (\log x)_{,\eta} = \frac{c\sqrt{\delta}}{AB\sqrt{\Delta}}\,\phi_{,\mu} \qquad (\log x)_{,\mu} = \frac{c\sqrt{\Delta}}{AB\sqrt{\delta}}\,\phi_{,\eta}$$

Moreover, the R.H.S. for the equation for Ω read

$$(c^2-4)\,\frac{\phi_\eta \phi_\mu}{A^2B^2}, \qquad (c^2-4)\,\frac{1}{A^2B^2}\left(\Delta\,\phi_{,\eta}^2 + \delta\,\phi_{,\mu}^2\right).$$

So, the choice $c=2$ drops ϕ, x completely out of the equation for Ω: we have 3×3, Ω, A, B, Ω algebraically, eliminate it, two equations for the two gauge functions!

$$\frac{\mathring{A}}{A}\left[\frac{(\mathring{B}\sqrt{\delta})^\circ\sqrt{\delta}}{B}\right]^\circ - \frac{\mathring{B}}{B}\left[\frac{(\mathring{A}\sqrt{\Delta})^\circ\sqrt{\Delta}}{A}\right]^\circ = \frac{\mathring{A}\mathring{B}}{AB}\left[\frac{(\mathring{B}\sqrt{\delta})^\circ\sqrt{\delta}}{B} - \frac{(\mathring{A}\sqrt{\Delta})^\circ\sqrt{\Delta}}{A}\right]$$

$$\frac{\mathring{A}\Delta}{A}\left[\frac{(\mathring{A}\sqrt{\Delta})^\circ\sqrt{\Delta}}{A}\right]^\circ - \frac{\mathring{B}\delta}{B}\left[\frac{(\mathring{B}\sqrt{\delta})^\circ\sqrt{\delta}}{B}\right]^\circ = \left[\frac{(\mathring{A}\sqrt{\Delta})^\circ\sqrt{\Delta}}{A} - \frac{(\mathring{B}\sqrt{\delta})^\circ\sqrt{\delta}}{B}\right]\times$$

$$\times\left\{\Delta\left(\frac{\mathring{\mathring{A}}}{A} - \frac{\mathring{A}^2}{2A^2}\right) + \delta\left(\frac{\mathring{\mathring{B}}}{B} - \frac{\mathring{B}^2}{2B^2}\right) - \left(\eta\frac{\mathring{A}}{A} + \mu\frac{\mathring{B}}{B}\right)\right\}$$

weak point?

Both third order, very non-linear equations. But $A = A(\eta)$, $B = B(\mu)$, so the system is over determined and it can be solved. Three families, I will show you one,

$$A = \frac{1}{\cos^2(c\arcsin\eta + d_1)}, \qquad B = \frac{1}{\cos^2(c\arcsin\mu + d_2)}$$

Then Ω algebraically from A, B, then solve the linear equation for ϕ (it admits simple separable solutions) then determine $(\log x)$ by quadratures. Or, solve linear equation for $(\log x)$, the gravitational field and determine the fluid (ϕ) by quadratures.

The situation is exactly the same as for the Weyl family of static solutions. We have a linear equation (Laplace there, hyperbolic here), we know how to solve (simple solutions by separation of variables) and for each particular case we have to decide which solutions of the linear equations to choose.

The solution will give ($\log x$) and the remaining of the problem can be completed by quadratures. But the strategy is different: For Weyl first $\log x$, then the conformal factor, now the conformal factor is determined ab initio, the same for the entire family, they $\log x$ and quadratures for the specification of the fluid.

Remarks:

(i) Actually, three families of solutions. Also,

$$A = \frac{1}{sh^2(c \arcsin \eta + d_1)}, \quad B = \frac{1}{sh^2(c \arcsin \mu + d_2)}$$

$$A = \frac{1}{(d_1 - \arcsin \eta)^2}, \quad B = \frac{1}{(d_2 - \arcsin \mu)^2}.$$

They can be mapped one to the other but for ranges of coordinates which do not overlap, they just cover different intervals. So, are they one or three families?

(ii) c, d_1, d_2 are not free parameters, they can be gauged out if you go to two-geometry $\sim ((dx)^2 - (dy)^2)$. They merely represent different ways you can then go to $\frac{(d\eta)^2}{1-\eta^2} - \frac{(d\mu)^2}{1-\mu^2}$. But I want to keep them because for particular values of them I get some very simple gauge functions (remember the complicated equations they had to satisfy).

Reason, $\cos^2(c \arcsin\eta + d_i)$ for c multiplier of $1/2$, $d_i = 0, \pm\frac{\pi}{4}, \pm\frac{\pi}{2}$ we get algebraic, even rational function.
Examples

$$\left(A = \frac{1}{1\pm\eta},\ B = \frac{1}{1\pm\mu}\right);\ \left(A = \frac{1}{1\pm\eta},\ B = \frac{1}{1\pm\sqrt{1-\mu^2}}\right),$$

$$\left(A = \frac{1}{1\pm\sqrt{1-\eta^2}},\ B = \frac{1}{1\pm\sqrt{1-\mu^2}}\right);\ \left(A = \frac{1}{\eta^2},\ B = \frac{1}{\mu^2}\right)$$

$$\left(A = \frac{1}{1-\eta^2},\ B = \frac{1}{1-\mu^2}\right);\ \left(A = \frac{1}{\eta^2(1-\eta^2)},\ B = \frac{1}{(1-2\mu^2)^2}\right)$$

symmetrical and asymmetrical choices because c the same but d_1 and d_2 different.

(iii) Very simple solution, $x = c\arcsin\eta + d_1$, $y = c\arcsin\mu + d_2$,
$\Phi \sim \dfrac{\cos k_0\frac{x}{2}}{\cos\frac{x}{2}}\ \dfrac{\cos(k_0\frac{y}{2})}{\cos\frac{y}{2}}$, not complete investigation, depends on for what you want to use them.

(iv) Although I have not worked out systematically through the different particular solutions (take my word, it does not, as usually, means that I've ~~cannot~~ tried and I cannot make it to work; Really I have not gone through systematically) we can already draw some general ~~solutions~~ conclusions if we are going to use these solutions for the description of the interaction regim of colliding gravitational and hydromagnetic waves.

The big question I am currently interested is whether colliding plane waves in Gen. Relativity focus or not. Old story, they focus on three-dimensional spacelike surfaces, new story, much less, on two dimensional time like surfaces? which is the prevailing mode of focusing?

And are they the only ones or other possibilities may arise?

The present family of solutions points towards a third posibility.

The general form of the metric is simple enough so that we can easily set up a Neuman–Penrose null tetrad and evaluate the Weyl and the Ricci scalars. They would depend on the gauge functions A, B, the conformal factor Ω, and norm of the Killing field α. Now we know A, B, Ω universally, for the entire family and we can simplify. Ψ_2 simplifies considerably

$$\Psi_2 \sim \frac{(\text{solution})}{(\text{family})}$$

and it predicts curvature singularities on null surfaces. Possibly you can eliminate them by chosing suitably the particular solutions but if you do get some, they will be on <u>null surfaces</u>, $u = \text{cont}$, $v = \text{cont}$, not necessarily $u = 1$ or $v = 1$. Certainly I have to do more work on that.

✓ A slight disadvantage: We started with $\epsilon = p + \kappa$ but eventually κ can be absorbed to a gauge and the only crucial is whether $k = 0, +1$ or -1. So, the present family is ~~too~~ not continuously connected to the $\epsilon = p$ solutions, so we cannot follow and see how the change in the place of focusing occurs.

(vi) For stationary axisymmetric spacetimes I expect a similar theory to hold for $p = \text{const}$ equation of state. So, it is not very much interesting for stellar interiors, except if you want to approximate stars with layers of constant pressure. But also we still do not have rotation.

Future:
- Again surface orthogonal and other simple equations of state
- What it gives for ~~...~~ inhomogeneous cosmological models and for colliding waves
- Non-hypersurface orthogonal Killing fields, something like rigid rotation. $u^{(4)}/u^{(t)} = \text{const}$.

But the ~~most~~ really interesting result, for me, was the conviction that we may even have to change strategy:
First the conformal factor and then the norm (and the twist) of the Killing field.

(vii) By assuming $G = c\phi$, we do not have the general solution of the $E = p + k$ equations, but just a big enough family (or three families).

$$ds^2 = e^{2\nu}(dx^0)^2 - e^{2\psi}(dx^1)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2$$

$(\partial/\partial x^1)^a$, $(\partial/\partial x^2)^a$: Killing fields.

$$T_{ab} = (\epsilon + p)\, u_a u_b - p\, g_{ab} \quad , \quad u^a u_a = 1.$$

$$\nabla^a T_{ab} = 0.$$

Equation of state :

$$\epsilon + p = f(\epsilon)$$

$$\frac{\dot f_1}{f_1} = \frac{1}{f}$$

$$\Big[\underbrace{e^{\nu+\mu_2+\mu_3}\, u_{(0)} f_1}_{\phi_{,3}}\Big]_{,0} - \Big[\underbrace{e^{\nu+\psi+\mu_2}\, u_{(3)} f_1}_{\phi_{,0}}\Big]_{,3} = 0$$

$$\Big[e^{\mu_3-\mu_2-\psi-\nu}\,\phi_{,0}\Big]_{,0} - \Big[e^{\nu-\psi-\mu_2-\mu_3}\,\phi_{,3}\Big]_{,3} = 0.$$

for $\epsilon = p + k$

like wave equation

$$2\epsilon - k = e^{-2(\psi + \mu_2)} \left[\phi_{,3}^{\,2}\, e^{-2\mu_3} - \phi_{,0}^{\,2}\, e^{-2\nu} \right]$$

$$u_{(0)} = \frac{e^{-\psi - \mu_2 - \mu_3}}{\sqrt{2\epsilon - k}} \; \phi_{,3}$$

$$u_{(3)} = \frac{e^{-\psi - \nu - \mu_2}}{\sqrt{2\epsilon - k}} \; \phi_{,0}$$

$$ds^2 = e^{2\nu}(dx^0)^2 - e^{2\psi}(dx^1)^2 - e^{2\mu_2}(dx^2)^2 - e^{2\mu_3}(dx^3)^2$$

$$\beta = \psi + \mu_2 \,, \qquad \chi = e^{\mu_2 - \psi}$$

Gauge

$$\left[e^{\mu_3-\nu}(e^\beta)_{,0}\right]_{,0} - \left[e^{\nu-\mu_3}(e^\beta)_{,3}\right]_{,3} = -2\kappa e^{\beta+\nu+\mu_3}$$

Norm

$$\left[e^{\beta+\mu_3-\nu}(\log\chi)_{,0}\right]_{,0} - \left[e^{\beta+\nu-\mu_3}(\log\chi)_{,3}\right]_{,3} = 0$$

conformal factor

$$\beta_{,0}(\nu+\mu_3)_{,3} + \beta_{,3}(\nu+\mu_3)_{,0} + \Big[\beta_{,0}(\nu-\mu_3)_{,3} - \beta_{,3}(\nu-\mu_3)_{,0} - $$
$$ -2\beta_{,03} - \beta_{,0}\beta_{,3}\Big] =$$

$$= \frac{\chi_{,0}\chi_{,3}}{\chi^2} - 4(\epsilon+p)\,u_{(0)}u_{(3)}\,e^{\nu+\mu_3}$$

$$2\beta_{,3}e^{\nu-\mu_3}(\nu+\mu_3)_{,3} + 2\beta_{,0}e^{\mu_3-\nu}(\nu+\mu_3)_{,0} + [\cdots\cdots] =$$
$$= e^{\nu-\mu_3}(\log\chi)_{,3}^2 + e^{\mu_3-\nu}(\log\chi)_{,0}^2 -$$
$$-4(\epsilon+p)\left[u_{(0)}^2 + u_{(3)}^2\right]e^{\nu+\mu_3}$$

fluid

$$\left[e^{\mu_3-\mu_2-\psi-\nu}\phi_{,0}\right]_{,0} - \left[e^{\nu-\psi-\mu_2-\mu_3}\phi_{,3}\right]_{,3} = 0$$

Gauge conditions on

$$\beta = \psi + \mu_2 \quad , \quad \mu_3 - \nu .$$

Vacuum,
Einstein-Maxwell :
$\epsilon = p$

$$e^{\beta} = \sqrt{(1-\eta^2)(1-\mu^2)}$$
$$e^{\mu_3 - \nu} = \sqrt{1-\eta^2}$$
$$\eta \sim x^0, \quad \mu \sim x^3$$

Present family :

$$e^{\beta} = A(x^0)\, B(x^3)$$
$$e^{\mu_3 - \nu} = \frac{\Gamma(x^0)}{E(x^3)}$$

$$\frac{dx^0}{\Gamma(x^0)} = \frac{d\eta}{\sqrt{1-\eta^2}} \quad ; \quad \frac{dx^3}{E(x^3)} = \frac{d\mu}{\sqrt{1-\mu^2}}$$

$$\begin{pmatrix}\text{two-dim.}\\ \text{geometry}\end{pmatrix} = e^{2\Omega}\left[\frac{(d\eta)^2}{1-\eta^2} - \frac{(d\mu)^2}{1-\mu^2}\right]$$

Metric:

$$ds^2 = e^{2\Omega}\left[\frac{(d\eta)^2}{1-\eta^2} - \frac{(d\mu)^2}{1-\mu^2}\right] -$$

$$- A(\eta)B(\mu)\left[\frac{(dx^1)^2}{x} + x\,(dx^2)^2\right]$$

Fluid:

$$2\epsilon - k = \frac{e^{-2\Omega}}{A^2B^2}\left(\delta\,\phi_{,\mu}^{\;2} - \Delta\,\phi_{,\eta}^{\;2}\right)$$

$$u_{(0)} = \frac{\phi_{,\mu}\sqrt{\delta}}{\sqrt{\delta\,\phi_{,\mu}^{\;2} - \Delta\,\phi_{,\eta}^{\;2}}}$$

$$u_{(3)} = \frac{\phi_{,\eta}\sqrt{\Delta}}{\sqrt{\delta\,\phi_{,\mu}^{\;2} - \Delta\,\phi_{,\eta}^{\;2}}}$$

$$\Delta = 1-\eta^2, \qquad \delta = 1-\mu^2.$$

Five equations – Five unknowns

$\boxed{\phi,\ \chi,\ \Omega,\ A,\ B}$ $\quad A(\eta),\ B(\mu)$

$$A\left[\frac{\sqrt{\Delta}}{A}\phi_{,\eta}\right]_{,\eta}\sqrt{\Delta} - B\left[\frac{\sqrt{\delta}}{B}\phi_{,\mu}\right]_{,\mu}\sqrt{\delta} = 0$$

$$\frac{1}{A}\left[A\sqrt{\Delta}\,(\log\chi)_{,\eta}\right]_{,\eta}\sqrt{\Delta} - \frac{1}{B}\left[B\sqrt{\delta}\,(\log\chi)_{,\mu}\right]_{,\mu}\sqrt{\delta} = 0$$

$$\frac{(\dot{A}\sqrt{\Delta})^{\cdot}\sqrt{\Delta}}{A} - \frac{(\dot{B}\sqrt{\delta})^{\cdot}\sqrt{\delta}}{B} = -2\kappa e^{2\Omega}$$

$$\frac{2\dot{A}}{A}\Omega_{,\mu} + \frac{2\dot{B}}{B}\Omega_{,\eta} = \frac{\dot{A}\dot{B}}{AB} + \boxed{\cancel{\frac{\chi_{,\eta}\chi_{,\mu}}{\chi^2}} - \cancel{\frac{4\phi_{,\eta}\phi_{,\mu}}{A^2B^2}}}$$

$$\frac{4\dot{A}\Delta}{A}\Omega_{,\eta} + \frac{4\dot{B}\delta}{B}\Omega_{,\mu} = \Delta\left(\frac{2\ddot{A}}{A} - \frac{\dot{A}^2}{A^2}\right) +$$

$$+\,\delta\left(\frac{2\ddot{B}}{B} - \frac{\dot{B}^2}{B^2}\right) - 2\left(\eta\frac{\dot{A}}{A} + \mu\frac{\dot{B}}{B}\right) +$$

$$\boxed{+\cancel{\frac{1}{\chi^2}\left(\Delta\chi_{,\eta}^2 + \delta\chi_{,\mu}^2\right)} - \cancel{\frac{4}{A^2B^2}\left(\Delta\phi_{,\eta}^2 + \delta\phi_{,\mu}^2\right)}}$$

$\Delta = 1-\eta^2;\quad \delta = 1-\mu^2.$

SURPRISE !

Total divergence:

$$\left[\frac{AB\sqrt{\Delta}}{\sqrt{\delta}}(\log x)_{,\eta}\right]_{,\eta} = \left[\frac{AB\sqrt{\delta}}{\sqrt{\Delta}}(\log x)_{,\mu}\right]_{,\mu}$$

Potential:

$$\underbrace{\frac{AB\sqrt{\Delta}}{\sqrt{\delta}}(\log x)_{,\eta}}_{G_{,\mu}} \qquad \underbrace{\frac{AB\sqrt{\delta}}{\sqrt{\Delta}}(\log x)_{,\mu}}_{G_{,\eta}}$$

$$(\log x)_{,\eta} = \frac{\sqrt{\delta}}{AB\sqrt{\Delta}}\, G_{,\mu} \quad ; \quad (\log x)_{,\mu} = \frac{\sqrt{\Delta}}{AB\sqrt{\delta}}\, G_{,\eta}$$

Integrability:

$$A\left[\frac{\sqrt{\Delta}}{A}\, G_{,\eta}\right]_{,\eta}\sqrt{\Delta} - B\left[\frac{\sqrt{\delta}}{B}\, G_{,\mu}\right]_{,\mu}\sqrt{\delta} = 0.$$

The equation for the stream potential ϕ!

$$G = c\,\phi\ ;$$

$$\left.\begin{aligned}(\log x)_{,\eta} &= \frac{c\sqrt{\delta}}{AB\sqrt{\Delta}}\ \phi_{,\mu}\\ (\log x)_{,\mu} &= \frac{c\sqrt{\Delta}}{AB\sqrt{\delta}}\ \phi_{,\eta}\end{aligned}\right\}$$

$$(c^2-4)\ \frac{\phi_{,\eta}\,\phi_{,\mu}}{A^2B^2}$$

$$(c^2-4)\ \frac{1}{A^2B^2}\left(\Delta\,\phi_{,\eta}^2 + \delta\ \phi_{,\mu}^2\right)$$

choise $\qquad c = \pm 2.$

$$\frac{\dot{A}}{A}\left[\frac{(\dot{B}\sqrt{\delta})^{\cdot}\sqrt{\delta}}{B}\right]^{\cdot}-\frac{\dot{B}}{B}\left[\frac{(\dot{A}\sqrt{\Delta})^{\cdot}\sqrt{\Delta}}{A}\right]^{\cdot}=$$

$$=\frac{\dot{A}\dot{B}}{AB}\left[\frac{(\dot{B}\sqrt{\delta})^{\cdot}\sqrt{\delta}}{B}-\frac{(\dot{A}\sqrt{\Delta})^{\cdot}\sqrt{\Delta}}{A}\right]$$

$$\frac{\dot{A}\Delta}{A}\left[\frac{(\dot{A}\sqrt{\Delta})^{\cdot}\sqrt{\Delta}}{A}\right]^{\cdot}-\frac{\dot{B}\delta}{B}\left[\frac{(\dot{B}\sqrt{\delta})^{\cdot}\sqrt{\delta}}{B}\right]^{\cdot}=$$

$$=\left[\frac{(\dot{A}\sqrt{\Delta})^{\cdot}\sqrt{\Delta}}{A}-\frac{(\dot{B}\sqrt{\delta})^{\cdot}\sqrt{\delta}}{B}\right]\times$$

$$\times\left\{\Delta\left(\frac{\ddot{A}}{A}-\frac{\dot{A}^2}{2A^2}\right)+\delta\left(\frac{\ddot{B}}{B}-\frac{\dot{B}^2}{2B^2}\right)-\left(\eta\frac{\dot{A}}{A}+\mu\frac{\dot{B}}{B}\right)\right\}$$

$\Delta = 1-\eta^2$; $\delta = 1-\mu^2$;

$A = A(\eta)$; $B = B(\mu)$

OVERDETERMINED !

$$A = \frac{1}{\cos^2(c \arcsin \eta + \lambda_1)} \quad ; \quad B = \frac{1}{\cos^2(c \arcsin \mu + \lambda_2)}$$

Simple choices:

$$\left(\frac{1}{1 \pm \eta}, \frac{1}{1 \pm \mu} \right) \; ; \; \left(\frac{1}{1 \pm \eta}, \frac{1}{1 \pm \sqrt{1-\mu^2}} \right)$$

$$\left(\frac{1}{\eta^2}, \frac{1}{\mu^2} \right) \; ; \; \left(\frac{1}{1-\eta^2}, \frac{1}{1-\mu^2} \right)$$

$$\left(\frac{1}{\eta^2}, \frac{1}{1-\mu^2} \right) ; \; \left(\frac{1}{\eta^2(1-\eta^2)}, \frac{1}{(1-2\mu^2)^2} \right)$$

$$A = \frac{1}{\mathrm{sh}^2(c \arcsin \eta + \lambda_1)} \quad , \quad B = \frac{1}{\mathrm{sh}^2(c \arcsin \eta + \lambda_2)}$$

$$A = \frac{1}{(\lambda_1 - \arcsin \eta)^2} \quad ; \quad B = \frac{1}{(\lambda_2 - \arcsin \mu)^2}$$