

(I)

The electromagnetic field in a
Schwarzschild Black-hole.

4x4 η_{mn} : $\det \eta_{mn} \neq 0$, $g_{\mu\nu} = \eta_{mn} e^m{}_{\mu} e^{\nu}$

Source: $J_{\mu} = [pc^2, -J_x, -J_y, -J_z] \rightarrow J_m = e_m{}^{\mu} J_{\mu}$

Unknown: $F_{\mu\nu} \rightarrow F_{mn} = e_m{}^{\mu} e_n{}^{\nu} F_{\mu\nu}$

Determine: $\Gamma_{mnp} = e_{m\mu} e_n{}^{\nu} e_p{}^{\lambda} \Gamma_{\mu\nu\lambda}$

$C_{mnp} = \Gamma_{mpn} - \Gamma_{mnp}$

Maxwell:

Tensor
form :

$$\left\{ \begin{aligned} F^{\mu\nu}{}_{; \nu} &= \frac{4\pi}{c} J^{\mu} \\ F_{\mu\nu}{}_{; \lambda} + F_{\nu\lambda}{}_{; \mu} + F_{\lambda\mu}{}_{; \nu} &= 0 \end{aligned} \right.$$

Tetrad
formalism :

$$\left\{ \begin{aligned} F_{mn}{}_{; \nu} e^{\nu} + F_{pn} \Gamma^p{}_{m\nu} + F_{mn} \Gamma^m{}_{p\nu} &= \frac{4\pi}{c} J_m \\ F_{rs, \lambda} e_t{}^{\lambda} + F_{st, \lambda} e_r{}^{\lambda} + F_{tr, \lambda} e_s{}^{\lambda} + \\ + F_{rm} C^m{}_{ts} + F_{sm} C^m{}_{rt} + F_{tm} C^m{}_{sr} &= 0 \end{aligned} \right.$$

Schwarzschild

(u, r, θ, ϕ)

$$u = t - \frac{r}{c} - \frac{r_s}{c} \ln\left(\frac{r-r_s}{r_s}\right)$$

$$g_{\mu\nu} = \begin{bmatrix} \left(1 - \frac{r_s}{r}\right) c^2 & & & & \\ & c & & & \\ & & 0 & & \\ & & & 0 & -r^2 \\ & & & & 0 \\ & & & & & -r^2 \sin^2 \theta \end{bmatrix}$$

II

choose: $\eta_{mn} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{bmatrix}$

Newman-Penrose
1962

$$\left\{ \begin{array}{l} e^0{}_{\mu} = [c, 0, 0, 0] \\ e^1{}_{\mu} = \left[\frac{c}{2} \left(1 - \frac{r_s}{r}\right), 1, 0, 0 \right] \\ e^2{}_{\mu} = \left[0, 0, \frac{-r}{\sqrt{2}}, \frac{-ir \sin \theta}{\sqrt{2}} \right] \\ e^3{}_{\mu} = \left[0, 0, \frac{-r}{\sqrt{2}}, \frac{ir \sin \theta}{\sqrt{2}} \right] \end{array} \right.$$

Define operators:

$$D = e^0{}_{\mu} \frac{\partial}{\partial x^{\mu}} = \frac{\partial}{\partial r}$$

$$\Delta = e^1{}_{\mu} \frac{\partial}{\partial x^{\mu}} = c^{-1} \frac{\partial}{\partial u} - \frac{1}{2} \left(1 - \frac{r_s}{r}\right) \frac{\partial}{\partial r}$$

$$\delta = e^2{}_{\mu} \frac{\partial}{\partial x^{\mu}} = \frac{1}{r} \mathcal{D} = \frac{1}{r} \left(\frac{1}{\sqrt{2}} \frac{\partial}{\partial \theta} + \frac{i}{\sqrt{2} \sin \theta} \frac{\partial}{\partial \phi} \right)$$

$$\bar{\delta} = \frac{1}{r} \bar{\mathcal{D}} = \text{complex conjugate}$$

(III)

Calculate $\Gamma_{\mu\nu\rho} \rightarrow C_{\mu\nu\rho} \rightarrow$

$$\begin{array}{l} 4 \text{ equations} \\ \text{unknowns} : \end{array} \left\{ \begin{array}{l} F_{23} - F_{01} = 2\phi_1 \\ F_{02} = \phi_2 \\ F_{13} = -\phi_0. \end{array} \right.$$

Second order uncoupled:

$$\begin{aligned} & \left[\left(\Delta - \frac{1}{2r} \right) (rD + 2) - \frac{1}{r} \left(\bar{\mathcal{D}} + \frac{\cot\theta}{\sqrt{2}} \right) \mathcal{D} \right] \Phi_1 = \\ & = \frac{2\pi}{c} \left[\left(\Delta - \frac{1}{2r} \right) (rJ_1) + \left(\bar{\mathcal{D}} + \frac{\cot\theta}{\sqrt{2}} \right) J_3 \right], \end{aligned}$$

similar for ϕ_0, ϕ_2 .

How can we come back?

$$\left. \begin{array}{l} F_{01} = -2 \operatorname{Re}(\phi_1) \\ F_{02} = \phi_2 \\ F_{03} = \phi_2^* \\ F_{12} = -\phi_0^* \\ F_{13} = -\phi_0 \\ F_{23} = 2i \operatorname{Im}(\phi_1) \end{array} \right\} \Rightarrow$$

$$F_{\mu\nu} = F_{mn} e^m{}_{\mu} e^n{}_{\nu}$$

Angular dependence:

$$\phi_1 : Y_{lm}$$

$$\phi_0 : Y^{\perp}_{lm}$$

$$\phi_2 : Y^{\perp}_{lm}$$

Static problem - test point charge.

$$\rho(\vec{r}, t) = q(u^0)^{-1} \delta(\vec{r} - \vec{r}'), \quad J^k = \rho(\vec{r}, t) u^k \rightarrow$$

$$J_m, \quad J_2 = J_3 = 0.$$

radial: $r(r-r_s) \frac{d^2 R_{\perp lm}}{dr^2} + (4r-3r_s) \frac{dR_{\perp lm}}{dr} + [2-l(l+1)] R_{\perp lm} =$

$$= 2\pi q \left[(r-r_s) \frac{d}{dr} + 1 \right] \left(\frac{\delta(r-r')}{r} \right) Y^{\perp}_{lm}(\theta', \phi')$$

Interior - exterior solutions, match $r=r'$.

(I) $r' \rightarrow r_s$, $\phi_0^{(e)}, \phi_1^{(e)}, \phi_2^{(e)}, \phi_1^{(i)} \rightarrow 0$

$$\lim \phi_0^{(i)} \neq 0, \quad \lim \phi_2^{(i)} \neq 0.$$

(II) $r \rightarrow r_s, r' = \text{const.} > r_s$ $\phi_1, \phi_2 \rightarrow 0.$

$$\phi_0 \rightarrow \text{finite!!}$$

Homogeneous, $r = r_s \cdot x \rightsquigarrow$

$$R_{0lm} \approx \frac{d}{dx} P_l(1-2x), \quad R_{1lm} \approx \frac{d}{dx} \left[(x-1) \frac{d}{dx} P_l(1-2x) \right],$$

$$R_{2lm} \approx \frac{x-1}{x} \frac{d}{dx} P_l(1-2x), \quad Q_l \text{ for exterior.}$$

$$\text{For } r=r_s : \quad R' = r^{l-1}, \quad R'' = r^{-l-2}$$

$$\neq 0 \left\{ \begin{aligned} R_{1lm}^{(i)}(r, r_s) &= (-1)^l \frac{(l-1)!(l-1)! r_s^{l+1}}{(2l)!} \frac{d}{dr} \left[\left(\frac{r}{r_s} - 1 \right) \frac{d P_l \left(1 - \frac{2r}{r_s} \right)}{dr} \right] \\ R_{1lm}^{(e)}(r, r_s) &= \frac{(-1)^{l+1} (2l+2)!}{[(l+1)!]^2 (l+1) r_s^l} \frac{d}{dr} \left[\left(\frac{r}{r_s} - 1 \right) \frac{d Q_l \left(1 - \frac{2r}{r_s} \right)}{dr} \right] \end{aligned} \right.$$

Match for $r=r'$.

$$\Phi_0(r, \theta, \phi) = \begin{cases} 2\sqrt{2} \pi q (r' - r_s) \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{\sqrt{l(l+1)}}{2l+1} R_{0l}^{(e)}(r') R_{0l}^{(i)}(r) Y_{lm}^*(\theta', \phi') Y_{lm}^{\frac{1}{2}}(\theta, \phi), & r < r' \\ 2\sqrt{2} \pi q (r' - r_s) \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{\sqrt{l(l+1)}}{2l+1} R_{0l}^{(i)}(r') R_{0l}^{(e)}(r) Y_{lm}^*(\theta', \phi') Y_{lm}^{\frac{1}{2}}(\theta, \phi), & r > r' \end{cases}$$

static black hole.

- i) space time - geometry not $g_{\mu\nu}$, tetrad formalism (N.P 1962)
- ii) equations.

No contribution in the g curvature - geometry, test charge

as vectors, t -components,
How we lower - rise indices.

arbitrary, 10 equations, 16 unknowns.

"projections", $J_{\mu\nu}$, $F_{\mu\nu}$, ~~symmetric~~ antisymmetric,
 $F_{\mu\nu}$ invariant under coordinate transformations but not tetrad

Special choice $\gamma_{mn} \Rightarrow e^m_t$ null $\underline{e}_m^t \underline{e}_{mp} = 0$

e_0^t, e_1^t real, e_2^t, e_3^t complex conjugate.

Continuity eq. $J^t; t = 0 \Rightarrow J_{\mu\nu, \rho} e^{\mu t} + J_{\mu\nu} \Gamma^{\mu\rho}{}_{\rho} = 0.$

$$2\partial \left(\frac{\cot\theta}{\sqrt{2}} + \bar{\partial} \right) \gamma^t{}_{lm} = -l(l+1) \gamma^t{}_{lm}$$

$$2\bar{\partial} \left(\frac{\cot\theta}{\sqrt{2}} + \partial \right) \gamma^t{}_{lm} = -l(l+1) \gamma^t{}_{lm}$$

g invariant.

Static. g at \vec{r}' from $t = -\infty$

no time dependencies \rightarrow equations, ~~second~~ right hand side "almost zero",
i) Solve the homogeneous. $r = r_s \cdot x$, unitless

$$R_{02lm} \approx \frac{d}{dx} P_l(1-2x) \quad R_{12lm} \approx \frac{d}{dx} \left[(x-1) \frac{d}{dx} P_l(1-2x) \right],$$

$$R_{22lm} \approx \frac{x-1}{x} \frac{d}{dx} P_l(1-2x), \quad \partial_l \text{ for } \alpha \text{ second solution.}$$

$$\text{Abel-Liouville} \Rightarrow W[R_0^{(i)}, R_0^{(e)}] = \frac{-\cancel{2}(2\ell+1)}{r^2(r-r_s)^2}$$

$$W[R_1^{(i)}, R_1^{(e)}] = \frac{-(2\ell+1)}{r^3(r-r_s)}, \quad W[R_2^{(i)}, R_2^{(e)}] = \frac{-(2\ell+1)}{r^4}.$$

$$R_{\text{obsv}}(r) = \begin{cases} A_\ell R_{\text{obsv}}^{(i)}(r), & r < r' \\ B_\ell R_{\text{obsv}}^{(e)}(r), & r > r'. \end{cases} \quad \text{continuity at } r=r'.$$

incoming not radial wave. for a point at $r'=r_s$.