Iraklion, January 13, 1994.

Dear Chandra:

I would try to outline my proposal for finding the solution for the colliding gravitational and hydrodynamic waves with $p = \varepsilon$.

Any vacuum solution is described by a $Z = \chi + i q_2$ and a $(\nu + \mu_3)_{vac}$ and the inclusion of the fluid, according to our general theory, is described by functions Φ (stream potential) and f, and $\nu + \mu_3$ will change according to

$$(\nu + \mu_3) = (\nu + \mu_3)_{vac} + f.$$

Z satisfies the Ernst equation and $\nu + \mu_3$ is determined from Z via quadratures. All these hold in region I. Extending into region II by $u \to uH(u)$, $v \to vH(v)$ we will get $Z = Z(v)$ and $\nu + \mu_3 = (\nu + \mu_3)(v)$ only. The metric will be of the form that we have in our big paper, Section 8, Eq. (150) and we can certainly write down the conditions that the solution is flat — namely, to demand that $L = M = N = 0$. In fact I have already integrated these equations for the case when the two spacelike Killing fields are hypersurface orthogonal, i.e., when $q_2 = 0$. The relevant calculations are shown on pages 286-292. In these pages $B(v) = \chi(u=0, v)$ and the found expressions for B are in fact the boundary conditions that $\chi(u,v)$ should satisfy on the $u=0$ boundary. Lets consider the symmetrical

problem and demand the symmetrical, by interchanging u and v, boundary condition on the $v=0$ boundary.

Suppose that we have found such a solution for χ, with the correct boundary conditions, in region I. (I will return later on how to find it). Then we perform the quadratures and obtain $(\nu + \mu_3)_{vac.}$ in region I. The $f(u,v)$, of the fluid, is still unknown (and unspecified), but

$$\nu + \mu_3 = (\nu + \mu_3)_v + f(u,v)$$

is the correct $(\nu + \mu_3)(u,v)$ in region I. Let

$A(v) = (\nu + \mu_3)(u=0, v)$ is the value that it gets after the extension into region II,

$$A(v) = (\nu + \mu_3)_{vac.}(u=0, v) + f(u=0, v).$$

But for any $B(v)$ we have determined the associated $A(v)$ — pages 289 - 292 — so that the resulting metric in region II is flat. From the last expression we can calculate $f(u=0,v)$ which is the required boundary condition for f and from the equations passing from f to Φ we can evaluate the corresponding boundary conditions for Φ. Finally, from the general theory of Copson — what we learnt the last week — we can write down the solution for Φ inside region I and complete the solution. The only last test will be to ~~[struck]~~ verify the positivity of the energy density in region I. The general solution for ϕ will be

$$\phi(x_0,y_0) = \phi(1,1)\, v(1,1;x_0,y_0) +$$

$$+ \int_{y_0}^{1} \underbrace{\left[-v\phi_y + \frac{\phi v}{2(x+y)}\right]}_{x=1} dy + \int_{x_0}^{1} \underbrace{\left[-v\phi_x + \frac{\phi v}{2(x+y)}\right]}_{y=1} dx ,$$

which is the unique solution for any given boundary conditions, where

$$v(x,y;x_0,y_0) = \frac{(x_0+y_0)^{1/2}}{(x+y)^{1/2}}\, P_{1/2}\left(1 + 2\frac{(x-x_0)(y-y_0)}{(x+y)(x_0+y_0)}\right).$$

Let me explain now how to solve the boundary value problem for χ. It turns out that the same Copson equation – page 87, Eq. 2 – but with $\nu = +1/2$ should be used this time! The quantity χ satisfies the equation

$$\left[(1-\eta^2)\chi_{,\eta}\right]_{,\eta} - \left[(1-\mu^2)\chi_{,\mu}\right]_{,\mu} = 0$$

which in the (u,v) coordinates reads

$$(1-u^2-v^2)(\log\chi)_{,uv} - \left[u(\log\chi)_{,v} + v(\log\chi)_{,u}\right] = 0!!$$

Except from the different sign infront of the squared bracket, $\log\chi$ and ϕ satisfy

the same equation. Setting $x = 1-2u^2$, $y = 1-2v^2$, exactly as for ϕ – we get

$$(\log\chi)_{xy} + \frac{1}{2(x+y)}\left[(\log\chi)_{,x} + (\log\chi)_{,y}\right] = 0,$$

which is the Copson equation for $\nu = 1/2$. Fortunately, we've leant last week how to solve the boundary value problem for this equation.

In principle the entire problem is straightforward but in practice it might be difficult. It would depend on whether we can find a simple solution for χ satisfying the correct boundary conditions. If, on the other hand, we will have to use the integral expressions for $\log\chi$ – involving integrals of $P_{-1/2}$ this time, obtaining the solution will be extremely difficult, if not impossible. The big irony is that for the stream potential we had the bad" solution $\phi = k\log(1-u^2-v^2)$ which satisfies the correct boundary conditions. Unfortunately, with the change in the sign, it is not a solution for $(\log\chi)$. I think, I will take the pessimistic point of view.

It is obvious that we shall be using a lot of Copsons equation $u_{xy} + \frac{\nu}{x+y}(u_x+u_y) = 0$ and his result that

$$V(x,y;x_0,y_0) = \frac{(x+y)^{\nu}}{(x_0+y_0)^{\nu}} P_{-\nu}\left(1+2\frac{(x-x_0)(y-y_0)}{(x+y)(x_0+y_0)}\right)$$

is the corresponding Riemann-Green function. I enclose the pages 571-574 in which I have directly verified his claim. I do not get the P_{-m+i} that you mention in the phone. I thought that it might be a phase factor, if $P_{1/2}$ is not singled valued but I found in a book the claim that $P_{1/2}(x)$ is single valued and well behaved in the interval $(-1, +\infty)$; and all our considerations involve the interval $[1, \infty)$.

I also include pages 531-536 in which, by using the solution for ϕ we had so far - bad solutions for the colliding waves problem - I've obtained certain mathematical identities.

My proposal is to write a paper, about the general theory of colliding gravitational and ~~electro~~ hydrodynamic waves (with $p=\epsilon$), with what we have so far: the reduction of the equations, the introduction of the stream potential, the general discussion of how the metric and the Newman-Penrose quantities change by the inclusion of the quantity f which is the only change in the gravitational field due to the presence of the fluid, a general study of the

C^0 extensions into regions two, that the perfect fluid equations cannot be possibly satisfied into regions II and III and therefore that we should look for flat extensions, the detailed discussion of the boundary conditions in terms of ϕ and f, the general solution for ϕ - in terms of integrals - for any boundary conditions, the general considerations that I've mentioned in the beginning of this letter, the explanation why the simplest choice (your solution with Valeria) does not work, the mathematical identities in an appendix, e.t.c. I think that we have a lot of useful results about the general theory of colliding gravitational and hydrodynamic waves. I am certain that most of these - except the solution for ϕ and χ by the ~~[illegible]~~ Riemann-Green method - are already in the $(n-1)^{\text{th}}$ draft you mentioned in the phone and which I did not receive so far. In the mean time, I will be looking for a solution of the χ-equation with the correct boundary conditions and if we can ~~include~~ find one, we can include it as well.

Some other, unrelated things now:

- In our big paper there is a factor of "2" missing in equation (1). The last term should be $-e^{2\mu_2}(dx^2)^2$. Could you please take care of that in the proofs?

- I should plan a 2-3 weeks visit to Turkey, to collaborate with M. Gürses, probably on the complete integrability of the Einstein-Maxwell equations. Is there still the chance that you may come to Crete in the Spring? And your other trip for 2 or 3 weeks in Crete in August is it still one? Certainly, I would like them ~~to~~ both very much.
- Last October you asked me whether I would liked it to come again to Chicago to work with you, without an official visiting faculty appointment from the University. I would always loved to, and the only problem you would have to solve will be to find some kind of financial support to cover my living expenses there. I have no other plans for the summer and I will be eligible for a sabbatical leave next academic year for one ~~sem~~ semester. The last thing means that I can be away from Crete from January to September 86. Since you will be my first choice, I have not made any plans or requests, before asking you first.
- Will you please send me a copy of your talk in the Indian Academy of Sciences, about the motivation which drives the scientists?

- I think that in February I will have some time to think seriously about your collective works? Are there any additional instructions? Did M. Schwarzschild agree?

With best regards to your wife I remain

Yours Sincerely,

Basilis.