From the office of
S. CHANDRASEKHAR

September 24

Dear Basilis,

The enclosed response from Coxeter confirms your view. I have not however digested Sen's discussion of the geodesics. You may wish to examine them.

Best wishes

Chandra

UNIVERSITY OF TORONTO
TORONTO, CANADA
M5S 1A1

DEPARTMENT OF MATHEMATICS

Sept. 9 '86

Dear Chandra,

Thanks for your letter of last month. Yes, I am as busy as ever, and I am pleased to see that you are too. Not seeing how to throw light on your 'elementary' problem myself, I showed it to Dipak Sen, whose reply I enclose. Here, in turn, is a puzzle in pure mathematics for you:

$$\int_1^6 \frac{\operatorname{arc\,sec} t \, dt}{(t+2)\sqrt{(t+1)(t+3)}} + 2\int_1^6 \frac{\operatorname{arc\,sec} t \, dt}{(t+2)\sqrt{t+1}} = \frac{2}{15}\pi^2 .$$

I will be surprised (and delighted) if anyone can verify this precise result without appealing to geometry. I found it by dissecting a certain simplex, of known volume, in spherical 3-space, and expressing the volumes of the 3 pieces (2 congruent) in terms of Schläfli functions.

Let us put $\bar{t} = \psi$, $\bar{r} = x^2$, $\bar{\theta} = \theta$, $\bar{\phi} = x'$; then Chandrasekhar's first form of the metric becomes

$$\boxed{ds^2 = d\bar{t}^2 - \sin^2\bar{t}\, d\bar{r}^2 - d\bar{\theta}^2 - \sin^2\bar{\theta}\, d\bar{\phi}^2}$$

Chandrasekhar shows that the following transformation of coordinates (with its inverse):

$$t = t(\bar{t},\bar{r},\bar{\theta},\bar{\phi}) = \frac{\sin\bar{t}\cosh\bar{r}}{\cos\bar{t} - \sin\bar{t}\sinh\bar{r}}$$

$$r = r(\bar{t},\bar{r},\bar{\theta},\bar{\phi}) = \frac{1}{\cos\bar{t} - \sin\bar{t}\sinh\bar{r}}$$

$$\theta = \theta(\bar{t},\bar{r},\bar{\theta},\bar{\phi}) = \bar{\theta}$$

$$\phi = \phi(\bar{t},\bar{r},\bar{\theta},\bar{\phi}) = \bar{\phi}$$

transforms the above metric form to the conformally flat form (Chandrasekhar's second form)

$$\boxed{ds^2 = \frac{1}{r^2}\left(dt^2 - dr^2 - r^2 d\theta^2 - r^2\sin^2\theta\, d\phi^2\right)}$$

Since the angle coordinates are not changed, we may just as well consider the 2-dimensional situation:

$$\boxed{ds^2 = d\bar{t}^2 - \sin^2\bar{t}\, d\bar{r}^2 = \frac{1}{r^2}(dt^2 - dr^2)}$$

I donot think that the transformations themselves have any geometrical significance However, the metric seems to have interesting geometrical properties. It is similar to the hyperbolic - plane (Poincaré model) metric

$$ds^2 = \frac{1}{r^2}(dt^2 + dr^2)$$

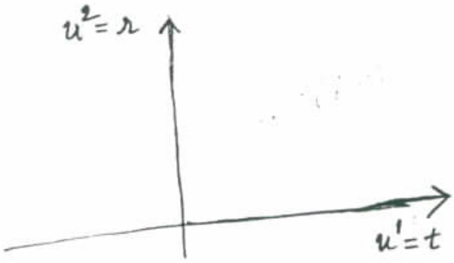
. Let us consider the geodesics of $ds^2 = \frac{1}{r^2}(dt^2 - dr^2)$

Geodesics: $ds^2 = \frac{1}{r^2}(dt^2 - dr^2)$

Singularity at $r=0$, so consider $r>0$. The metric is conformal to Minkowski metric $ds^2 = dt^2 - dr^2$. So null geodesics are same i.e. $t = \pm r$

For non-null geodesics consider

$$ds^2 = \frac{1}{r^2}(dt^2 - dr^2), \quad t = u^1, \; r = u^2$$



$$g_{11} = \frac{1}{r^2}, \quad g_{12} = 0, \quad g_{22} = -\frac{1}{r^2}$$

$$g^{11} = r^2, \quad g^{12} = 0, \quad g^{22} = -r^2$$

$$\Gamma^k_{ik} = g_{kk,i}/2g_{kk}, \quad \Gamma^k_{ii} = -g_{ii,k}/2g_{kk} \quad (i \neq k)$$

$$\Gamma^1_{11} = g_{11,1}/2g_{11} = 0 \qquad \Gamma^2_{12} = 0, \quad \Gamma^1_{22} = 0$$

$$\Gamma^2_{11} = -g_{11,2}/2g_{22} = -\left(\frac{1}{r^2}\right)' \Big/ 2\left(-\frac{1}{r^2}\right) = \frac{r^2}{2}\left(\frac{1}{r^2}\right)' = -1/r$$

$$\Gamma^2_{22} = g_{22,2}/2g_{22} = \left(-\frac{1}{r^2}\right)' \Big/ 2\left(-\frac{1}{r^2}\right) = -\frac{1}{r}$$

$$\Gamma^1_{21} = g_{11,2}/2g_{11} = \left(\frac{1}{r^2}\right)' \Big/ 2\left(\frac{1}{r^2}\right) = -1/r$$

$$\ddot{u}^k + \Gamma^k_{ij}\dot{u}^i\dot{u}^j = 0$$

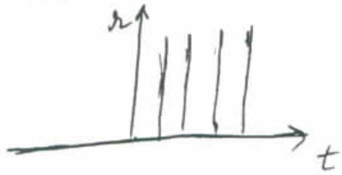
$$\ddot{u}^1 + \Gamma^1_{11}(\dot{u}^1)^2 + \Gamma^1_{22}(\dot{u}^2)^2 + 2\Gamma^1_{12}\dot{u}^1\dot{u}^2 = 0$$

$$\ddot{u}^1 - \frac{2\dot{u}^1\dot{u}^2}{r} = 0 \iff \boxed{\ddot{t} - \frac{2\dot{t}\dot{r}}{r} = 0}$$

$$\ddot{u}^2 + \Gamma^2_{11}(\dot{u}^1)^2 + \Gamma^2_{22}(\dot{u}^2)^2 + 2\Gamma^2_{12}\dot{u}^1\dot{u}^2 = 0$$

$$\ddot{u}^2 - \frac{(\dot{u}^1)^2}{r} - \frac{(\dot{u}^2)^2}{r} = 0 \iff \boxed{\ddot{r} - \frac{\dot{t}^2 + \dot{r}^2}{r} = 0}$$

If $\dot t = 0 \Rightarrow t = \text{const}$ is a soln. These are lines $\perp$ to $r = 0$



If $\dot t \neq 0 \Rightarrow \ddot t - \frac{2\dot t \dot r}{r} = 0 \Rightarrow \frac{\ddot t}{\dot t} - \frac{2\dot r}{r} = 0 \Rightarrow \boxed{\dot t = c r^2}$ $c = $ const

Because $\left[\ln\left(\frac{\dot t}{r^2}\right)\right]^{\cdot} = \frac{r^2}{\dot t}\left(\frac{\dot t}{r^2}\right)^{\cdot} = \frac{r^2}{\dot t}\left(\frac{\ddot t r^2 - 2r\dot r \dot t}{r^4}\right) = \frac{\ddot t}{\dot t} - \frac{2\dot r}{r} = 0$

Therefore $\ln\left(\frac{\dot t}{r^2}\right) = $ Const or $\boxed{\dot t = c r^2}$ $c = $ const

Now $\ddot r - \frac{\dot t^2 + \dot r^2}{r} = 0 \Rightarrow \ddot r \dot r r - \dot t^2 \dot r - \dot r^3 = 0 \Rightarrow \boxed{-\dot t^2 + \dot r^2 = b r^2}$ $b =$

Because $-\dot t^2 + \dot r^2 = b r^2 \Rightarrow -2\dot t \ddot t + 2\dot r \ddot r = b 2 r \dot r$

$\Rightarrow -\dot t \ddot t r + \ddot r \dot r r = \dot r b r^2 = \dot r(-\dot t^2 + \dot r^2) = -\dot t^2 \dot r + \dot r^3$

$\Rightarrow -\dot t \frac{2\dot t \dot r}{r} r + \ddot r \dot r r + \dot t^2 \dot r - \dot r^3 = 0$

or $\ddot r \dot r r - \dot t^2 \dot r - \dot r^3 = 0$

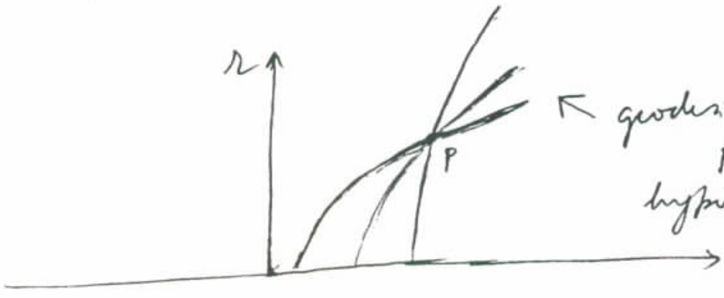
Therefore the geodesics are given by $\boxed{\dot t = c r^2}$ & $\boxed{-\dot t^2 + \dot r^2 = b r^2}$

So $\left(\frac{dr}{dt}\right)^2 = \frac{\dot r^2}{\dot t^2} = \frac{b}{c^2 r^2} + 1 \Rightarrow \boxed{(t-a)^2 - r^2 = b/c^2} \Leftrightarrow$ hyperbolas along t-axis

Because $(t-a)^2 - r^2 = b/c^2$

$\Rightarrow 2(t-a) - 2r\frac{dr}{dt} = 0$

or $\left(\frac{dr}{dt}\right)^2 = \frac{(t-a)^2}{r^2} = \frac{b}{c^2 r^2} + 1$



↖ geodesics thru P are hyperbolas

Note:- Compare with $ds^2 = \frac{1}{r^2}(dt^2 + dr^2)$, $r > 0$; which is the Hyperbolic plane (Poincaré model), where the geodesics are given by $(t-a)^2 + r^2 = b/c^2$, that is, circles with centers on $r = 0$